

# GEODESIC DISTANCE FOR RIGHT INVARIANT SOBOLEV METRICS OF FRACTIONAL ORDER ON THE DIFFEOMORPHISM GROUP. II

MARTIN BAUER, MARTINS BRUVERIS, PETER W. MICHOR

ABSTRACT. The geodesic distance vanishes on the group  $\text{Diff}_c(M)$  of compactly supported diffeomorphisms of a Riemannian manifold  $M$  of bounded geometry, for the right invariant weak Riemannian metric which is induced by the Sobolev metric  $H^s$  of order  $0 \leq s < \frac{1}{2}$  on the Lie algebra  $\mathfrak{X}_c(M)$  of vector fields with compact support.

## 1. INTRODUCTION

In the article [1] we studied right invariant metrics on the group  $\text{Diff}_c(M)$  of compactly supported diffeomorphisms of a manifold  $M$ , which are induced by the Sobolev metric  $H^s$  of order  $s$  on the Lie algebra  $\mathfrak{X}_c(M)$  of vector fields with compact support. We showed that for  $M = S^1$  the geodesic distance on  $\text{Diff}(S^1)$  vanishes if and only if  $s \leq \frac{1}{2}$ . For other manifolds, we showed that the geodesic distance on  $\text{Diff}_c(M)$  vanishes for  $M = \mathbb{R} \times N$ ,  $s < \frac{1}{2}$  and for  $M = S^1 \times N$ ,  $s \leq \frac{1}{2}$ , with  $N$  being a compact Riemannian manifold.

Now we are able to complement this result by: *The geodesic distance vanishes on  $\text{Diff}_c(M)$  for any Riemannian manifold  $M$  of bounded geometry, if  $0 \leq s < \frac{1}{2}$ .*

We believe that this result holds also for  $s = \frac{1}{2}$ , but we were able to overcome the technical difficulties only for the manifold  $M = S^1$ , in [1]. We also believe that it is true for the regular groups  $\text{Diff}_{\mathcal{H}^\infty}(\mathbb{R}^n)$  and  $\text{Diff}_{\mathcal{S}}(\mathbb{R}^n)$  as treated in [8], and for all Virasoro groups, where we could prove it only for  $s = 0$  in [2].

In Section 2, we review the definitions for Sobolev norms of fractional orders on diffeomorphism groups as presented in [1] and extend them to diffeomorphism groups of manifolds of bounded geometry. Section 3 is devoted to the main result.

## 2. SOBOLEV METRICS $H^s$ WITH $s \in \mathbb{R}$

**2.1. Sobolev metrics  $H^s$  on  $\mathbb{R}^n$ .** For  $s \geq 0$  the Sobolev  $H^s$ -norm of an  $\mathbb{R}^n$ -valued function  $f$  on  $\mathbb{R}^n$  is defined as

$$(1) \quad \|f\|_{H^s(\mathbb{R}^n)}^2 = \|\mathcal{F}^{-1}(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}f\|_{L^2(\mathbb{R}^n)}^2,$$

where  $\mathcal{F}$  is the Fourier transform

$$\mathcal{F}f(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} f(x) \, dx,$$

---

2010 *Mathematics Subject Classification.* Primary 35Q31, 58B20, 58D05.

Martin Bauer was supported by ‘Fonds zur Förderung der wissenschaftlichen Forschung, Projekt P 24625’.

and  $\xi$  is the independent variable in the frequency domain. An equivalent norm is given by

$$(2) \quad \|f\|_{H^s(\mathbb{R}^n)}^2 = \|f\|_{L^2(\mathbb{R}^n)}^2 + \| |\xi|^s \mathcal{F}f \|_{L^2(\mathbb{R}^n)}^2 .$$

The fact that both norms are equivalent is based on the inequality

$$\frac{1}{C} \left(1 + \sum_j |\xi_j|^s\right) \leq \left(1 + \sum_j |\xi_j|^2\right)^{\frac{s}{2}} \leq C \left(1 + \sum_j |\xi_j|^s\right) ,$$

holding for some constant  $C$ . For  $s > 1$  this says that all  $\ell^s$ -norms on  $\mathbb{R}^{n+1}$  are equivalent. But the inequality is true also for  $0 < s < 1$ , even though the expression does not define a norm on  $\mathbb{R}^{n+1}$ . Using any of these norms we obtain the Sobolev spaces with non-integral  $s$

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \|f\|_{H^s(\mathbb{R}^n)} < \infty\} .$$

We will use the second version of the norm in the proof of the theorem, since it will make calculations easier.

## 2.2. Sobolev metrics for Riemannian manifolds of bounded geometry.

Following [13, Section 7.2.1] we will now introduce the spaces  $H^s(M)$  on a manifold  $M$ . If  $M$  is not compact we equip  $M$  with a Riemannian metric  $g$  of bounded geometry which exists by [5]. This means that

- (I) The injectivity radius of  $(M, g)$  is positive.
- ( $B_\infty$ ) Each iterated covariant derivative of the curvature is uniformly  $g$ -bounded:  $\|\nabla^i R\|_g < C_i$  for  $i = 0, 1, 2, \dots$ .

The following is a compilation of special cases of results collected in [3, Chapter 1], who treats Sobolev spaces only for integral order.

**Proposition** ([6], [10], [4]). *If  $(M, g)$  satisfies (I) and ( $B_\infty$ ) then the following holds:*

- (1)  $(M, g)$  is complete.
- (2) There exists  $\varepsilon_0 > 0$  such that for each  $\varepsilon \in (0, \varepsilon_0)$  there is a countable cover of  $M$  by geodesic balls  $B_\varepsilon(x_\alpha)$  such that the cover of  $M$  by the balls  $B_{2\varepsilon}(x_\alpha)$  is still uniformly locally finite.
- (3) Moreover, there exists a partition of unity  $1 = \sum_\alpha \rho_\alpha$  on  $M$  such that  $\rho_\alpha \geq 0$ ,  $\rho_\alpha \in C_c^\infty(M)$ ,  $\text{supp}(\rho_\alpha) \subset B_{2\varepsilon}(x_\alpha)$ , and  $|D_u^\beta \rho_\alpha| < C_\beta$  where  $u$  are normal (Riemann exponential) coordinates in  $B_{2\varepsilon}(x_\alpha)$ .
- (4) In each  $B_{2\varepsilon}(x_\alpha)$ , in normal coordinates, we have  $|D_u^\beta g_{ij}| < C'_\beta$ ,  $|D_u^\beta g^{ij}| < C''_\beta$ , and  $|D_u^\beta \Gamma_{ij}^m| < C'''_\beta$ , where all constants are independent of  $\alpha$ .

We can now define the  $H^s$ -norm of a function  $f$  on  $M$ :

$$\begin{aligned} \|f\|_{H^s(M, g)}^2 &= \sum_{\alpha=0}^{\infty} \|(\rho_\alpha f) \circ \exp_{x_\alpha}\|_{H^s(\mathbb{R}^n)}^2 = \\ &= \sum_{\alpha=0}^{\infty} \|\mathcal{F}^{-1}(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}((\rho_\alpha f) \circ \exp_{x_\alpha})\|_{L^2(\mathbb{R}^n)}^2 . \end{aligned}$$

If  $M$  is compact the sum is finite. Changing the charts or the partition of unity leads to equivalent norms by the proposition above, see [13, Theorem 7.2.3]. For integer  $s$  we get norms which are equivalent to the Sobolev norms treated in [3,

Chapter 2]. The norms depends on the choice of the Riemann metric  $g$ . This dependence is worked out in detail in [3].

For vector fields we use the trivialization of the tangent bundle that is induced by the coordinate charts and define the norm in each coordinate as above. This leads to a (up to equivalence) well-defined  $H^s$ -norm on the Lie algebra  $\mathfrak{X}_c(M)$ .

**2.3. Sobolev metrics on  $\text{Diff}_c(M)$ .** Given a norm on  $\mathfrak{X}_c(M)$  we can use the right-multiplication in the diffeomorphism group  $\text{Diff}_c(M)$  to extend this norm to a right-invariant Riemannian metric on  $\text{Diff}_c(M)$ . In detail, given  $\varphi \in \text{Diff}_c(M)$  and  $X, Y \in T_\varphi \text{Diff}_c(M)$  we define

$$G_\varphi^s(X, Y) = \langle X \circ \varphi^{-1}, Y \circ \varphi^{-1} \rangle_{H^s(M)} .$$

We are interested solely in questions of vanishing and non-vanishing of geodesic distance. These properties are invariant under changes to equivalent inner products, since equivalent inner products on the Lie Algebra

$$\frac{1}{C} \langle X, Y \rangle_1 \leq \langle X, Y \rangle_2 \leq C \langle X, Y \rangle_1$$

imply that the geodesic distances will be equivalent metrics

$$\frac{1}{C} \text{dist}_1(\varphi, \psi) \leq \text{dist}_2(\varphi, \psi) \leq C \text{dist}_1(\varphi, \psi) .$$

Therefore the ambiguity – dependence on the charts and the partition of unity – in the definition of the  $H^s$ -norm is of no concern to us.

### 3. VANISHING GEODESIC DISTANCE

**3.1. Theorem** (Vanishing geodesic distance). *The Sobolev metric of order  $s$  induces vanishing geodesic distance on  $\text{Diff}_c(M)$  if:*

- $0 \leq s < \frac{1}{2}$  and  $M$  is any Riemannian manifold of bounded geometry.

*This means that any two diffeomorphisms in the same connected component of  $\text{Diff}_c(M)$  can be connected by a path of arbitrarily short  $G^s$ -length.*

In the proof of the theorem we shall make use of the following lemma from [1].

**3.2. Lemma** ([1, Lemma 3.2]). *Let  $\varphi \in \text{Diff}_c(\mathbb{R})$  be a diffeomorphism satisfying  $\varphi(x) \geq x$  and let  $T > 0$  be fixed. Then for each  $0 \leq s < \frac{1}{2}$  and  $\varepsilon > 0$  there exists a time dependent vector field  $u_\mathbb{R}^\varepsilon$  of the form*

$$u_\mathbb{R}^\varepsilon(t, x) = \mathbb{1}_{[f^\varepsilon(t), g^\varepsilon(t)]} * G_\varepsilon(x),$$

*with  $f, g \in C^\infty([0, T])$ , such that its flow  $\varphi^\varepsilon(t, x)$  satisfies – independently of  $\varepsilon$  – the properties  $\varphi^\varepsilon(0, x) = x$ ,  $\varphi^\varepsilon(T, x) = \varphi(x)$  and whose  $H^s$ -length is smaller than  $\varepsilon$ , i.e.,*

$$\text{Len}(\varphi^\varepsilon) = \int_0^T \|u_\mathbb{R}^\varepsilon(t, \cdot)\|_{H^s} dt \leq C \|f^\varepsilon - g^\varepsilon\|_\infty \leq \varepsilon .$$

*Furthermore  $\{t : f^\varepsilon(t) < g^\varepsilon(t)\} \subseteq \text{supp}(\varphi)$  and there exists a limit function  $h \in C^\infty([0, T])$ , such that  $f^\varepsilon \rightarrow h$  and  $g^\varepsilon \rightarrow h$  for  $\varepsilon \rightarrow 0$  and the convergence is uniform in  $t$ .*

Here,  $G_\varepsilon(x) = \frac{1}{\varepsilon} G_1(\frac{x}{\varepsilon})$  is a smoothing kernel, defined via a smooth bump function  $G_1$  with compact support.

*Proof of Theorem 3.1.* Consider the connected component  $\text{Diff}_0(M)$  of  $\text{Id}$ , i.e. those diffeomorphisms of  $\text{Diff}_c(M)$ , for which there exists at least one path, joining them to the identity. Denote by  $\text{Diff}_c(M)^{L=0}$  the set of all diffeomorphisms  $\varphi$  that can be reached from the identity by curves of arbitrarily short length, i.e., for each  $\varepsilon > 0$  there exists a curve from  $\text{Id}$  to  $\varphi$  with length smaller than  $\varepsilon$ .

**Claim A.**  $\text{Diff}_c(M)^{L=0}$  is a normal subgroup of  $\text{Diff}_0(M)$ .

**Claim B.**  $\text{Diff}_c(M)^{L=0}$  is a non-trivial subgroup of  $\text{Diff}_0(M)$ .

By [12] or [7], the group  $\text{Diff}_0(M)$  is simple. Thus claims A and B imply  $\text{Diff}_c(M)^{L=0} = \text{Diff}_0(M)$ , which proves the theorem.

The proof of claim A can be found in [1, Theorem 3.1] and works without change in the case of  $M$  being an arbitrary manifold and hence we will not repeat it here. It remains to show that  $\text{Diff}_c(M)^{L=0}$  contains a diffeomorphism  $\varphi \neq \text{Id}$ .

We shall first prove claim B for  $M = \mathbb{R}^n$  and then show how to extend the arguments to arbitrary manifolds. Choose a diffeomorphism  $\varphi_{\mathbb{R}} \in \text{Diff}_c(\mathbb{R})$  with  $\text{supp}(\varphi_{\mathbb{R}}) \subseteq [1, \infty)$  and let

$$u_{\mathbb{R}}^{\varepsilon}(t, x) := \mathbb{1}_{[f^{\varepsilon}(t), g^{\varepsilon}(t)]} * G_{\varepsilon}(x)$$

be the family of vector fields constructed in Lemma 3.2, whose flows at time  $T$  equal  $\varphi_{\mathbb{R}}$ . We extend the vector field  $u_{\mathbb{R}}^{\varepsilon}$  to a vector field  $u_{\mathbb{R}^n}^{\varepsilon}$  on  $\mathbb{R}^n$  via

$$u_{\mathbb{R}^n}^{\varepsilon}(x_1, \dots, x_n) := (u_{\mathbb{R}}^{\varepsilon}(|x|), 0, \dots, 0) .$$

The flow of this vector field is given by

$$\varphi_{\mathbb{R}^n}^{\varepsilon}(t, x_1, \dots, x_n) = (\varphi_{\mathbb{R}}^{\varepsilon}(t, |x|), x_2, \dots, x_n) ,$$

where  $\varphi_{\mathbb{R}}^{\varepsilon}$  is the flow of  $u_{\mathbb{R}}^{\varepsilon}$ . In particular we see that at time  $t = T$

$$\varphi_{\mathbb{R}^n}^{\varepsilon}(t, x_1, \dots, x_n) = (\varphi_{\mathbb{R}}(|x|), x_2, \dots, x_n) ,$$

the flow is independent of  $\varepsilon$ . So it remains to show that for the length of the path  $\varphi_{\mathbb{R}^n}^{\varepsilon}(t, \cdot)$  we have

$$\text{Len}(\varphi_{\mathbb{R}^n}^{\varepsilon}) \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0 .$$

We can estimate the length of this path via

$$\begin{aligned} \text{Len}(\varphi_{\mathbb{R}^n}^{\varepsilon})^2 &= \left( \int_0^T \|u_{\mathbb{R}^n}^{\varepsilon}(t, \cdot)\|_{H^s(\mathbb{R}^n)} dt \right)^2 \leq T \int_0^T \|u_{\mathbb{R}^n}^{\varepsilon}(t, \cdot)\|_{H^s(\mathbb{R}^n)}^2 dt \\ &= T \int_0^T \|u_{\mathbb{R}}^{\varepsilon}(t, |\cdot|)\|_{H^s(\mathbb{R})}^2 dt = T \int_0^T \|\mathbb{1}_{[f^{\varepsilon}(t), g^{\varepsilon}(t)]} * G_{\varepsilon}(|x|)\|_{H^s(\mathbb{R})}^2 dt \\ &\leq C(G_1, T) \int_0^T \|\mathbb{1}_{[f^{\varepsilon}(t), g^{\varepsilon}(t)]}(|\cdot|)\|_{H^s(\mathbb{R})}^2 dt , \end{aligned}$$

where the last estimate follows from

$$\begin{aligned} \|\mathbb{1}_{[f^{\varepsilon}(t), g^{\varepsilon}(t)]} * G_{\varepsilon}(|x|)\|_{H^s(\mathbb{R})}^2 &= \\ &= \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) [\mathcal{F}(\mathbb{1}_{[f^{\varepsilon}(t), g^{\varepsilon}(t)]}(|\cdot|))(\xi)]^2 [\mathcal{F}(G_{\varepsilon}(|\cdot|))(\xi)]^2 d\xi \\ &= \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) [\mathcal{F}(\mathbb{1}_{[f^{\varepsilon}(t), g^{\varepsilon}(t)]}(|\cdot|))(\xi)]^2 [\mathcal{F}(G_1(|\cdot|))(\varepsilon\xi)]^2 d\xi \\ &\leq \|\mathcal{F}G_1(|\cdot|)\|_{L^{\infty}}^2 \cdot \|\mathbb{1}_{[f^{\varepsilon}(t), g^{\varepsilon}(t)]}(|\cdot|)\|_{H^s(\mathbb{R})}^2 . \end{aligned}$$

Hence it is sufficient to show that

$$\|\mathbb{1}_{[f^\varepsilon(t), g^\varepsilon(t)]}(|\cdot|)\|_{H^s(\mathbb{R}^n)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad \text{uniformly in } t.$$

To compute the  $H^s$ -norm of  $\mathbb{1}_{[f^\varepsilon(t), g^\varepsilon(t)]}(|\cdot|)$  we first Fourier-transform it. The Fourier-transform of a radially symmetric function  $v(|\cdot|) \in L^1(\mathbb{R}^n)$  is again radially symmetric and given by the following formula, see [11, Theorem 3.3],

$$(\mathcal{F}v(|\cdot|))(\xi) = 2\pi|\xi|^{1-n/2} \int_0^\infty J_{n/2-1}(2\pi|\xi|s)v(s)s^{n/2} ds,$$

with  $J_{n/2-1}$  denoting the Bessel function of order  $\frac{n}{2} - 1$ . To simplify notation we will omit the dependence of the vector field  $\mathbb{1}_{[f^\varepsilon(t), g^\varepsilon(t)]}(|\cdot|)$  on  $t$  and  $\varepsilon$ . Changing coordinates, this becomes

$$(\mathcal{F}\mathbb{1}_{[f,g]}(|\cdot|))(\xi) = (2\pi)^{-n/2}|\xi|^{-n} \int_{2\pi f|\xi|}^{2\pi g|\xi|} J_{n/2-1}(s)s^{n/2} ds.$$

This integral can be evaluated explicitly using the following integral identity for Bessel functions from [9, (10.22.1)]

$$\int z^{\nu+1} J_\nu(z) dz = z^{\nu+1} J_{\nu+1}(z), \quad \nu \neq -\frac{1}{2}.$$

This gives us

$$(\mathcal{F}\mathbb{1}_{[f,g]}(|\cdot|))(\xi) = |\xi|^{-n/2} \left( J_{n/2}(2\pi g|\xi|)g^{n/2} - J_{n/2}(2\pi f|\xi|)f^{n/2} \right).$$

The  $H^s$ -norm of  $\mathbb{1}_{[f,g]}(|\cdot|)$  is given by

$$\|\mathbb{1}_{[f,g]}(|\cdot|)\|_{H^s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) \mathcal{F}\mathbb{1}_{[f,g]}(|\cdot|)(\xi)^2 d\xi.$$

We will only consider the term involving  $|\xi|^{2s}$ , since the  $L^2$ -term can be estimated in the same way by setting  $s = 0$ . Transforming to polar coordinates we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |\xi|^{2s} (\mathcal{F}\mathbb{1}_{[f,g]}(|\cdot|)(\xi))^2 d\xi &= \\ &= \int_{\mathbb{R}^n} |\xi|^{2s-n} \left( J_{n/2}(2\pi g|\xi|)g^{n/2} - J_{n/2}(2\pi f|\xi|)f^{n/2} \right)^2 d\xi \\ &= \text{Vol}(S^{n-1}) \int_0^\infty r^{2s-1} \left( J_{n/2}(2\pi gr)g^{n/2} - J_{n/2}(2\pi fr)f^{n/2} \right)^2 dr. \end{aligned}$$

The above integral is non-zero only for those  $t$ , where  $f^\varepsilon(t) \neq g^\varepsilon(t)$ . From Lemma 3.2 and our assumptions on  $\varphi_{\mathbb{R}}$  we know that

$$\{t : f^\varepsilon(t) < g^\varepsilon(t)\} \subseteq \text{supp}(\varphi_{\mathbb{R}}) \subseteq [1, \infty).$$

Thus both  $f^\varepsilon(t)$  and  $g^\varepsilon(t)$  are different and away from 0 and we can evaluate the above integral using the identity [9, (10.22.57)],

$$\int_0^\infty \frac{J_\mu(at)J_\nu(at)}{t^\lambda} dt = \frac{(\frac{1}{2}a)^{\lambda-1} \Gamma(\frac{\mu}{2} + \frac{\nu}{2} - \frac{\lambda}{2} + \frac{1}{2}) \Gamma(\lambda)}{2\Gamma(\frac{\lambda}{2} + \frac{\nu}{2} - \frac{\mu}{2} + \frac{1}{2}) \Gamma(\frac{\lambda}{2} + \frac{\mu}{2} - \frac{\nu}{2} + \frac{1}{2}) \Gamma(\frac{\lambda}{2} + \frac{\mu}{2} + \frac{\nu}{2} + \frac{1}{2})},$$

which holds for  $\operatorname{Re}(\mu + \nu + 1) > \operatorname{Re} \lambda > 0$  and the identity [9, (10.22.56)],

$$\begin{aligned} \int_0^\infty \frac{J_\mu(at)J_\nu(bt)}{t^\lambda} dt &= \\ &= \frac{a^\mu \Gamma\left(\frac{\nu}{2} + \frac{\mu}{2} - \frac{\lambda}{2} + \frac{1}{2}\right)}{2^\lambda b^{\mu-\lambda+1} \Gamma\left(\frac{\nu}{2} - \frac{\mu}{2} + \frac{\lambda}{2} + \frac{1}{2}\right)} \mathbf{F}\left(\frac{\nu}{2} + \frac{\mu}{2} - \frac{\lambda}{2} + \frac{1}{2}, \frac{\mu}{2} - \frac{\nu}{2} - \frac{\lambda}{2} + \frac{1}{2}; \mu + 1; \frac{a^2}{b^2}\right), \end{aligned}$$

which holds for  $0 < a < b$  and  $\operatorname{Re}(\mu + \nu + 1) > \operatorname{Re} \lambda > -1$ . Here  $\mathbf{F}(a, b; c; d)$  is the regularized hypergeometric function. Using these identities with  $\lambda = 1 - 2s$ ,  $\mu = \nu = \frac{n}{2}$ ,  $a = 2\pi f$  and  $b = 2\pi g$  we obtain

$$\int_0^\infty r^{2s-1} J_{n/2}(2\pi fr)^2 dr = \frac{1}{2} (\pi f)^{-2s} \frac{\Gamma\left(\frac{n}{2} + s\right) \Gamma(1 - 2s)}{\Gamma(1 - s)^2 \Gamma\left(\frac{n}{2} + 1 - s\right)}$$

and

$$\begin{aligned} \int_0^\infty r^{2s-1} J_{n/2}(2\pi fr) J_{n/2}(2\pi gr) dr &= \\ &= \frac{1}{2} (\pi g)^{-2s} \left(\frac{f}{g}\right)^{n/2} \frac{\Gamma\left(\frac{n}{2} + s\right)}{\Gamma(1 - s)} \mathbf{F}\left(\frac{n}{2} + s, s; \frac{n}{2} + 1; \frac{f^2}{g^2}\right). \end{aligned}$$

Putting it together results in

$$\begin{aligned} \int_{\mathbb{R}^n} |\xi|^{2s} (\mathcal{F} \mathbb{1}_{[f,g]}(|\cdot|))(\xi)^2 d\xi &= \\ &= \operatorname{Vol}(S^{n-1}) \left( \frac{f^{-2s} + g^{-2s}}{2\pi^{2s}} \frac{\Gamma\left(\frac{n}{2} + s\right) \Gamma(1 - 2s)}{\Gamma(1 - s)^2 \Gamma\left(\frac{n}{2} + 1 - s\right)} - \right. \\ &\quad \left. - \frac{g^{-2s}}{\pi^{2s}} \frac{f^{n/2}}{g^{n/2}} \frac{\Gamma\left(\frac{n}{2} + s\right)}{\Gamma(1 - s)} \mathbf{F}\left(\frac{n}{2} + s, s; \frac{n}{2} + 1; \frac{f^2}{g^2}\right) \right). \end{aligned}$$

In the limit  $\varepsilon \rightarrow 0$  we know from Lemma 3.2 that  $f^\varepsilon(t) \rightarrow h(t)$  and  $g^\varepsilon(t) \rightarrow h(t)$  uniformly in  $t$  on  $[0, T]$  and hence  $\frac{f^\varepsilon(t)}{g^\varepsilon(t)} \rightarrow 1$ . For the regularized hypergeometric function  $\mathbf{F}(a, b; c; d)$  at  $d = 1$  we have the identity [9, (15.4.20)]

$$\mathbf{F}(a, b; c; 1) = \frac{\Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)},$$

for  $\operatorname{Re}(c - a - b) > 0$ . Applying the identity with  $a = \frac{n}{2} + s$ ,  $b = s$  and  $c = \frac{n}{2} + 1$  we get

$$\mathbf{F}\left(\frac{n}{2} + s, s; \frac{n}{2} + 1; 1\right) = \frac{\Gamma(1 - 2s)}{\Gamma(1 - s) \Gamma\left(\frac{n}{2} + 1 - s\right)}.$$

Using the continuity of the hypergeometric function it follows that

$$\int_{\mathbb{R}^n} |\xi|^{2s} (\mathcal{F} \mathbb{1}_{[f,g]}(|\cdot|))(\xi)^2 d\xi \rightarrow 0,$$

as  $\varepsilon \rightarrow 0$  and the convergence is uniform in  $t$ . This concludes the proof that

$$\left\| \mathbb{1}_{[f^\varepsilon(t), g^\varepsilon(t)]}(|\cdot|) \right\|_{H^s(\mathbb{R}^n)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad \text{uniformly in } t,$$

and hence we have established claim B for  $\operatorname{Diff}_c(\mathbb{R}^n)$ .

To prove this result for an arbitrary manifold  $M$  of bounded geometry we choose a partition of unity  $(\tau_j)$  such that  $\tau_0 \equiv 1$  on some open subset  $U \subset M$ , where normal coordinates centred at  $x_0 \in M$  are defined. If  $\varphi_{\mathbb{R}}$  is chosen with sufficiently

small support, then the vector field  $u_{\mathbb{R}^n}^\varepsilon$  has support in  $\exp_{x_0}(U)$  and we can define the vector field  $u_M^\varepsilon := (\exp_{x_0}^{-1})^* u_{\mathbb{R}^n}^\varepsilon$  on  $M$ . This vector field generates a path  $\varphi_M^\varepsilon(t, \cdot) \in \text{Diff}_0(M)$  with an endpoint  $\varphi_M^\varepsilon(T, \cdot) = \varphi_M(\cdot)$  that doesn't depend on  $\varepsilon$  with arbitrarily small  $H^s$ -length since

$$\begin{aligned} \text{Len}(\varphi_M^\varepsilon) &\leq C_1(\tau) \int_0^T \|u_M^\varepsilon\|_{H^s(M, \tau)} dt = C_1(\tau) \int_0^T \|\exp_{x_0}^*(\tau_0 \cdot u_M^\varepsilon)\|_{H^s(\mathbb{R}^n)} dt \\ &= C_1(\tau) \int_0^T \|u_{\mathbb{R}^n}^\varepsilon\|_{H^s(\mathbb{R}^n)} dt. \end{aligned}$$

Thus we can reduce the case of arbitrary manifolds to  $\mathbb{R}^n$  and this concludes the proof.  $\square$

## REFERENCES

- [1] Martin Bauer, Martins Bruveris, Philipp Harms, and Peter W. Michor. Geodesic distance for right invariant sobolev metrics of fractional order on the diffeomorphism group. *To appear in Ann. Global Anal. Geom.*, 2012.
- [2] Martin Bauer, Martins Bruveris, Philipp Harms, and Peter W. Michor. Vanishing geodesic distance for the Riemannian metric with geodesic equation the KdV-equation. *Ann. Global Anal. Geom.*, 41(4):461–472, 2012.
- [3] J. Eichhorn. *Global analysis on open manifolds*. Nova Science Publishers Inc., New York, 2007.
- [4] Jürgen Eichhorn. The boundedness of connection coefficients and their derivatives. *Math. Nachr.*, 152:145–158, 1991.
- [5] R. E. Greene. Complete metrics of bounded curvature on noncompact manifolds. *Arch. Math. (Basel)*, 31(1):89–95, 1978/79.
- [6] Yu. A. Kordyukov.  $L^p$ -theory of elliptic differential operators on manifolds of bounded geometry. *Acta Appl. Math.*, 23(3):223–260, 1991.
- [7] J. N. Mather. Commutators of diffeomorphisms. *Comment. Math. Helv.*, 49:512–528, 1974.
- [8] Peter W. Michor and David Mumford. A zoo of diffeomorphism groups on  $\mathbb{R}^n$ . 2012. In preparation.
- [9] Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark, editors. *NIST handbook of mathematical functions*. U.S. Department of Commerce National Institute of Standards and Technology, Washington, DC, 2010.
- [10] M. A. Shubin. Spectral theory of elliptic operators on noncompact manifolds. *Astérisque*, (207):5, 35–108, 1992. Méthodes semi-classiques, Vol. 1 (Nantes, 1991).
- [11] Elias M. Stein and Guido Weiss. *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.
- [12] W. Thurston. Foliations and groups of diffeomorphisms. *Bull. Amer. Math. Soc.*, 80:304–307, 1974.
- [13] Hans Triebel. *Theory of function spaces. II*, volume 84 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1992.

MARTIN BAUER, PETER W. MICHOR: FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT WIEN, NORDBERGSTRASSE 15, A-1090 WIEN, AUSTRIA.

MARTINS BRUVERIS: INSITUT DE MATHÉMATIQUES, EPFL, CH-1015, LAUSANNE, SWITZERLAND.

*E-mail address:* `bauer.martin@univie.ac.at`

*E-mail address:* `martins.bruveris@epfl.ch`

*E-mail address:* `peter.michor@esi.ac.at`